Shot noise displaying simultaneously the Noah and Joseph effects

Iddo Eliazar[1,2,*] and Joseph Klafter[2,†]
[1]Department of Technology Management, Holon Institute of Technology, Holon 58102, Israel
[2]School of Chemistry, Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel
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Shot noise processes displaying the Noah effect are of infinite variance. Hence the measurement of the correlation structure of such processes—indicating whether the Joseph effect is displayed as well—cannot be conducted via their autocovariance functions (which are undefined). To circumvent this problem, a Poisson-based analysis applicable to shot noise processes with infinite variance, as well as to shot noise processes with divergent noise levels, is developed. *A Poissonian autocovariance* function, which characterizes the process-distribution of general shot noise processes, is introduced. In particular, this function governs and quantifies the processes' stationary structure (amplitudal behavior) and correlation structure (temporal behavior). The "Poissonian methodology" developed enables a precise quantitative analysis of shot noise processes displaying simultaneously the Noah and Joseph effects.

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I. INTRODUCTION

Shot noise is the most fundamental quantitative model of *discontinuous* noise in continuous-time physical systems. The first documented theoretical works on shot noise were published in 1909 by Campbell [1,2]. In these works Campbell studied the discontinuous nature of light emission. The first documented physical observation of shot noise was reported in 1918 by Schottky, after having experimented with ideal vacuum tubes [3]. A comprehensive analysis of shot noise was conducted in the mid-1940s by Rice [4,5]. Modern "textbook" sources on shot noise include [6,7].

The discontinuous nature of shot noise stems from the discreteness of the "quantas" carrying its flow: be it photons in optical systems (e.g., Campbell's studies); electrons in electrical systems (e.g., Schottky's experiment); packets in digital communication systems; etc.

Coined by Mandelbrot and Wallis, the *Noah effect* and the *Joseph effect* are used to term "fractal behavior" of stationary random processes [8]. The former indicates "amplitudal fractality"—characterized by stationary distributions with asymptotically power-law probability tails; the latter indicates "temporal fractality"—characterized by autocovariance functions with asymptotically power-law tails.

"Classic" shot noise processes can display the Noah effect, but cannot display the Joseph effect (this is since the autocorrelation function of "classic" shot noise is exponential). On the other hand, generalizations of "classic" shot noise—namely *linear* shot noise processes [9,10], and *nonlinear* shot noise processes [11–14]—can exhibit both the Noah and the Joseph effects.

The *simultaneous* study of the Noah and Joseph effects is, however, rather problematic. Processes exhibiting the Noah effect have infinite variances. Consequently, they fail to possess autocovariance functions–via which their temporal correlations are usually analyzed. The same fundamental problem arises also in the case of random processes driven by stable non-Gaussian Lévy noise sources [15].

In recent research, the temporal correlation structure of random processes driven by general infinite-variance Lévy noise sources was analyzed and characterized via the novel methodology of *Lévy correlation cascades* [16]. This methodology further enabled the study of the ergodic properties of the processes under consideration [17].

In this paper we apply the core-concepts of the methodology of Lévy correlation cascades to the study of shot noise. General shot noise processes—including, in particular, the aforementioned linear and nonlinear processes—possess an underlying Poissonian structure. Exploiting this underlying structure, a Poissonian-based statistical analysis of shot noise is developed.

A *Poissonian autocovariance* function is introduced. This function: (i) is easily computable and tractable; (ii) characterizes the process-distribution of the shot noise processes under consideration; and (iii) is always well-defined, even in cases of divergent noise levels (let alone noise levels of infinite variance).

In particular, the Poissonian autocovariance function governs and quantifies the shot noise's *stationary structure* (amplitudal behavior) and *correlation structure* (temporal behavior). Since it is always well-defined, the Poissonian autocovariance function facilitates a precise quantitative analysis of shot noise processes displaying, simultaneously, the Noah and Joseph effects, thus addressing our initial goal.

The paper is organized as follows: We begin, in Sec. II, with the modeling of general shot noise processes. The Poissonian autocovariance function is introduced in Sec. III, followed by an analysis of the underlying Poissonian structure of general shot noise processes. Based on the Poissonian analysis, the statistical properties of shot noise—including, in particular, the Noah effect and the Joseph effect—are investigated in Sec. IV. Deeper probabilistic interpretations of the Poissonian autocovariance are unveiled and explained in Sec. V.

*Email address: eliazar@post.tau.ac.il
†Email address: klafter@post.tau.ac.il

II. SHOT NOISE SYSTEMS

From an abstract perspective, shot noise is a general model for arbitrary dissipative systems perturbed by random external "pulses" or "shots." A generic shot noise system-model is concisely described by the following triplet of rules.

(1) Stochasticity: external shots of random magnitudes hit the system randomly in time.

(2) Dissipation: shot magnitudes, after hitting the system, decay to zero.

(3) Superposition: the overall effect of the shots on the system is additive. Namely, the system's noise level at time t is the *aggregate* of all shot-magnitudes present in the system at time t.

Let us now describe, in detail, the setting of the shot noise system-model to be explored.

A. Stochasticity

The system's "shot-inflow" is considered Poisson, and we denote by $\Lambda(l)$ the *Poissonian rate* at which shots of magnitude greater than the level l "hit" the system ($l>0$). In other words, the time periods elapsing between the occurrences of shots with magnitudes greater than the level l are independent random variables, exponentially distributed with mean $1/\Lambda(l)$.

The function $\Lambda(\cdot)$ is monotone decreasing to zero: $\lim_{l\to\infty}\Lambda(l)=0$. The system's overall shot-inflow rate is $\Lambda(0) \coloneqq \lim_{l\to 0}\Lambda(l)$, which may be either finite or infinite.

Traditionally, the system's shot-inflow is taken to be a compound Poisson process. Namely, it is assumed that: (i) shots hit the system according to a standard Poisson process; and (ii) shot magnitudes are independent and identically distributed random variables (independent of the Poisson arrival process). This implies that the rate function $\Lambda(\cdot)$ is *bounded* $[\Lambda(0)<\infty]$ and that there are only *finitely* many shots hitting the system during any given time interval. More specifically, if λ denotes the rate of the standard Poisson process, and $\mathbf{P}_{>}(\cdot)$ denotes the survival probability of the shot magnitudes, then $\lambda=\Lambda(0)$ and $\mathbf{P}_{>}(l)=\Lambda(l)/\Lambda(0)$ ($l\geq 0$).

Considering *unbounded* rate functions $[\Lambda(0)=\infty]$ enable the system's overall shot-inflow rate to be *infinite*, this, in turn, implies that there are *infinitely* many shots hitting the system during any given time interval—a case which fails to be encompassed by compound Poisson inflows. We emphasize that *power-law* rate functions—which characterize *scale-invariant* shot-inflows—are always unbounded.

B. Dissipation

We denote by $\Phi(\tau;x)$ ($\tau,x\geq 0$) the shots' *decay pattern*. Namely, $\Phi(\tau;x)$ is the magnitude, τ units of time after "impact" of a shot with initial magnitude x. As a function of the "lag variable" τ the decay pattern $\Phi(\cdot;x)$ is assumed monotone decreasing from the initial shot magnitude $\Phi(0;x)=x$ to zero: $\lim_{\tau\to\infty}\Phi(\tau;x)=0$. As a function of the "magnitude variable" x the decay pattern $\Phi(\tau;\cdot)$ is assumed monotone increasing from $\Phi(\tau;0)=0$ to infinity: $\lim_{x\to\infty}\Phi(\tau;x)=\infty$.

We set $\Psi(\tau;y)$ ($\tau,y\geq 0$) to be the shots' *inverse decay pattern* with respect to the "magnitude variable" x. Namely, the function $\Psi(\tau;\cdot)$ is the inverse of the function $\Phi(\tau;\cdot)$ (the former is indeed well-defined since the latter is monotone increasing from zero to infinity).

Three important classes of shot noise system-models are as follows.

(1) Linear shot noise systems. In this class of systems the decay pattern is of the form $\Phi(\tau;x)=xh(\tau)$, where $h(\cdot)$ is an arbitrary *impulse-response function* decreasing monotonically from $h(0)=1$ to $\lim_{\tau\to\infty}h(\tau)=0$. The decay pattern is linear in the "magnitude variable" x. The inverse decay pattern is $\Psi(\tau;y)=y/h(\tau)$.

(2) $M/G/\infty$-type shot noise systems. In this class of systems the decay pattern is of the form $\Phi(\tau;x)=\max\{x-g(\tau),0\}$, where $g(\cdot)$ is an arbitrary *draining function* increasing monotonically from $g(0)=0$ to $\lim_{\tau\to\infty}g(\tau)=\infty$. The resulting shot noise processes are generalizations of the workload process of an $M/G/\infty$ queuing system—described in the Appendix (see Sec. A 1). The inverse decay pattern is $\Psi(\tau;y)=y+g(\tau)$.

(3) Nonlinear shot noise systems. In this class of systems the shots' decay-dynamics are governed by an arbitrary nonlinear ordinary differential equation (ODE) [11–14]. The decay pattern $\Phi(\tau;x)$ is the solution of the ODE, with initial condition x. The inverse decay pattern is $\Psi(\tau;y)=\Phi(-\tau;x)$—the *backwards* solution of the ODE, with terminal condition x.

Examples of the decay patterns of these classes are depicted in Figs. 1–4.

C. System processes

The three following system processes will play a key role in the sequel.

(1) The collection process $\Xi=[\Xi(t)]_t$, where $\Xi(t)$ denotes the collection of shot magnitudes present in the system at time t. The shot noise dissipation rule implies that a shot "hitting" the system at time $t'\leq t$, with initial magnitude x($x>0$), will contribute the magnitude $\Phi(t-t';x)$ to the collection $\Xi(t)$.

(2) The lth level process $N_l=[N_l(t)]_t$, where $N_l(t)$ denotes the number of shot-magnitudes, present in the system at time t, which are above the resolution level l ($l>0$). In other words, $N_l(t)$ denotes the cardinality (i.e., the size) of the collection $\Xi(t)\cap(l,\infty)$.

(3) The noise process $\xi=[\xi(t)]_t$, where $\xi(t)$ denotes the system's noise level at time t. The shot noise superposition rule implies that $\xi(t)=\sum_{y\in\Xi(t)}y$.

Schematic illustrations of these processes are depicted in Figs. 5–7. We note that the noise process ξ can be constructed directly from the level processes $\{N_l\}_{l>0}$ via

$$\xi(t)=\int_0^\infty N_l(t)dl.$$

III. ANALYSIS OF THE LEVEL PROCESSES

We analyze the general shot noise system-model via its *level processes* $\{N_l\}_{l>0}$. As shall be soon demonstrated, the

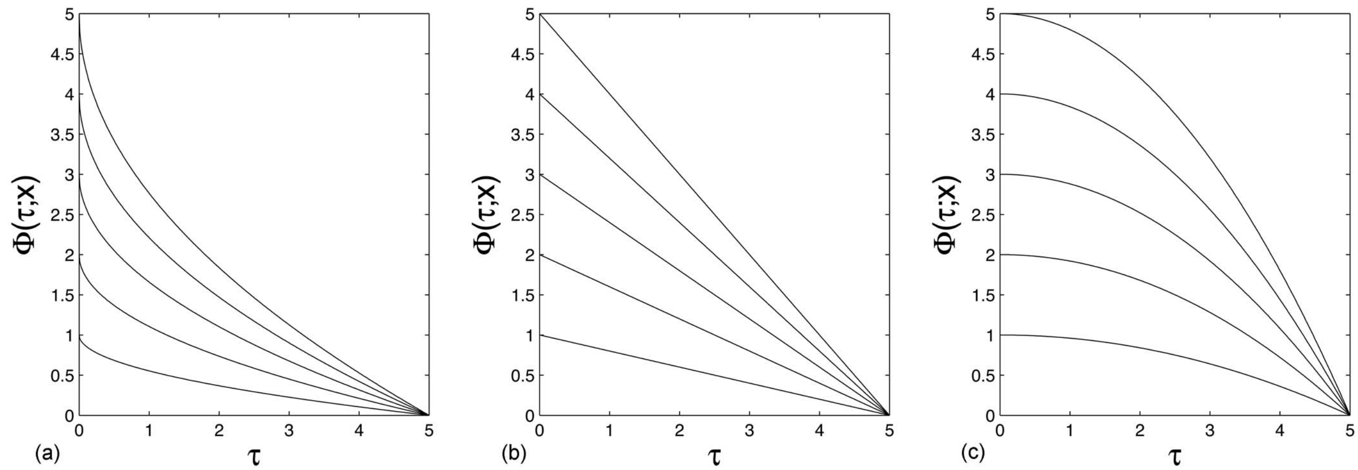

FIG. 1. Decay patterns of linear shot noise systems with impulse-response function $h(\tau)=\max\{1-(\tau/5)^p,0\}$ (p being a positive-valued parameter). (a) $p=0.5$; (b) $p=1$; and (c) $p=2$. The decay patterns are finite-ranged and convex for $p<1$; linear for $p=1$; and concave for $p>1$. Note that the "lifetimes" of the decay patterns are identical, irrespective of the size of the initial shot magnitudes.

noise's statistical behavior—both amplitudal and temporal—is governed by the function

$$F(\tau;l) = \int_{\tau}^{\infty} \Lambda(\Psi(s;l))ds \tag{1}$$

($\tau \geqslant 0$; $l>0$), which is contingent on the Poissonian rate function $\Lambda(\cdot)$ and on the inverse decay pattern $\Psi(\cdot;\cdot)$.

A. First-order and second-order analysis

The mean function, the autocovariance function, and the autocorrelation function of the lth level process N_l are given, respectively, by

(1) Mean:

$$\mu(l) \coloneqq \mathbf{E}[N_l(t)] = F(0;l). \tag{2}$$

(2) Autocovariance:

$$r_l(\tau) \coloneqq \mathrm{Cov}[N_l(t),N_l(t+\tau)] = F(\tau;l). \tag{3}$$

(3) Autocorrelation:

$$\rho_l(\tau) \coloneqq \mathrm{Cor}[N_l(t),N_l(t+\tau)] = \frac{F(\tau;l)}{F(0;l)}. \tag{4}$$

We henceforth coin these functions, respectively, the "Poissonian mean," the "Poissonian autocovariance," and the "Poissonian autocorrelation." Note that the Poissonian mean and the Poissonian autocorrelation are both induced by the function $F(\cdot;\cdot)$—the Poissonian autocovariance.

The proofs of Eqs. (2) and (3) are given in the Appendix (see Sec. A 2); Eq. (4) follows trivially from Eq. (3).

B. One-dimensional and two-dimensional marginal distributions

The Poissonian autocovariance was shown to govern the first-order and second-order statistics of the level processes

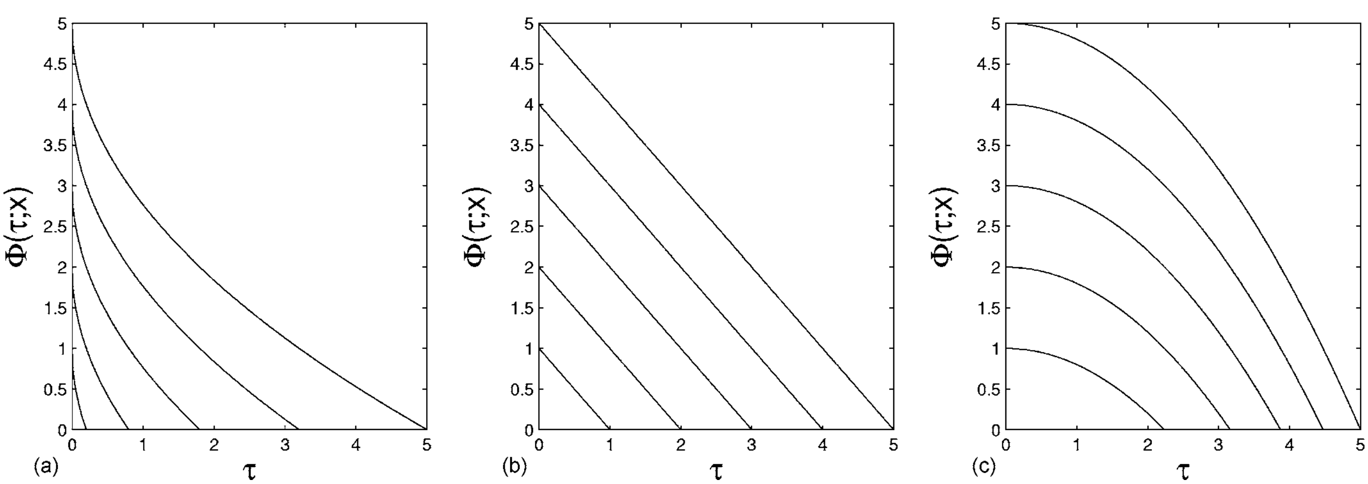

FIG. 2. Decay patterns of $M/G/\infty$-type shot noise systems with draining function $g(\tau)=5(\tau/5)^p$ (p being a positive-valued parameter). (a) $p=0.5$; (b) $p=1$; and (c) $p=2$. The decay patterns are finite-ranged and convex for $p<1$; linear for $p=1$; and concave for $p>1$. Note that the greater the initial shot magnitude, the longer the "lifetime" of the resulting decay pattern.

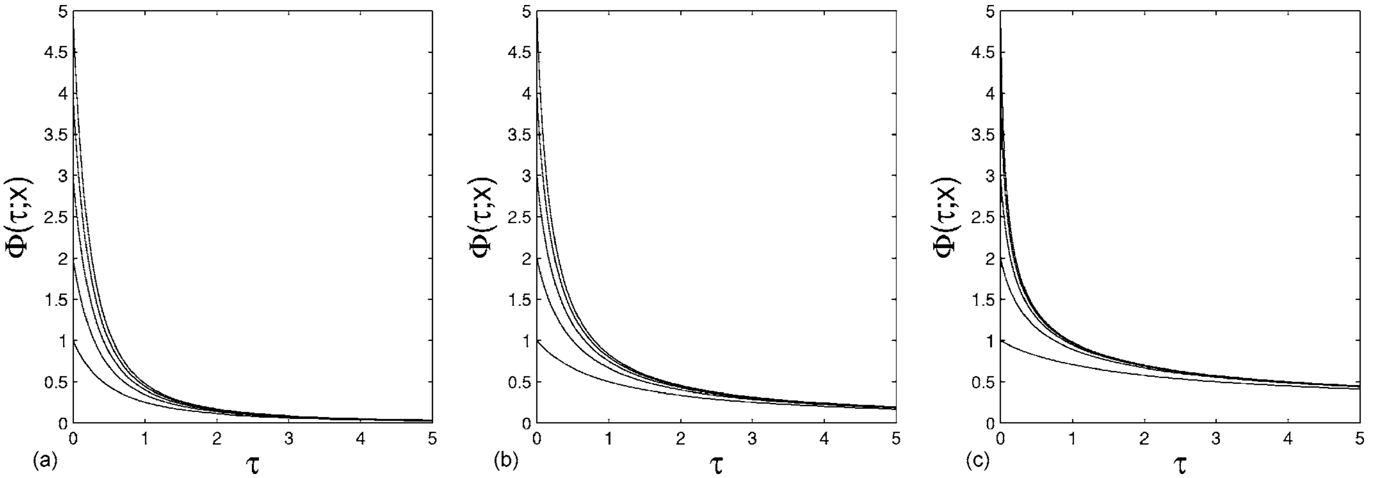

FIG. 3. Decay patterns of nonlinear shot noise systems with decay-dynamics governed by the ODE $\dot{X}=-(1/p)X^{1+p}$ (p being a positive-valued parameter). (a) $p=0.5$; (b) $p=1$; and (c) $p=2$. The decay patterns are infinite-ranged and convex.

$\{N_l\}_{l>0}$. In fact, this function *characterizes* the one-dimensional and two-dimensional *marginal distributions* of the level processes. Indeed, the corresponding one-dimensional and two-dimensional probability generating functions are given by

(1) One-dimensional:

$$\mathbf{E}[z^{N_l(t)}] = \exp\{\mu(l)(z-1)\} \quad (|z| \leqslant 1). \tag{5}$$

(2) Two-dimensional:

$$\mathbf{E}[z_1^{N_l(t)} z_2^{N_l(t+\tau)}] = \exp\{\mu(l)(z_1-1) + \mu(l)(z_2-1) + r_l(\tau)(z_1-1) \times (z_2-1)\} \quad (|z_i| \leqslant 1; i=1,2). \tag{6}$$

In particular, these results imply that the random variable $N_l(t)$ is *Poisson-distributed* with mean $\mu(l)$. Equations (5) and (6) are special cases of a more general result presented in the following section.

C. Multidimensional marginal distributions

The "reach" of the Poissonian autocovariance function goes far beyond what was demonstrated so far. Surprisingly, this function *characterizes all multidimensional marginal distributions* of the level processes $\{N_l\}_{l>0}$. As in the case of the one-dimensional and the two-dimensional marginal distributions, this assertion is ascertained via the corresponding multidimensional probability generating functions. The general result is as follows:

$$\mathbf{E}[z_1^{N_l(t_1)} \cdots z_n^{N_l(t_n)}] = \exp\{P_l(t_1, \ldots, t_n; z_1, \ldots, z_n)\} \tag{7}$$

($|z_i| \leqslant 1$; $i=1,\ldots,n$), where

$$P_l(t_1, \ldots, t_n; z_1, \ldots, z_n) = \sum_{k=1}^{n} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} r_l(\Delta(t_{i_1}, \ldots, t_{i_k})) \times (z_{i_1}-1) \cdots (z_{i_k}-1), \tag{8}$$

and where $\Delta(t_{i_1}, \ldots, t_{i_k}) := \max\{t_{i_1}, \ldots, t_{i_k}\} - \min\{t_{i_1}, \ldots, t_{i_k}\}$.

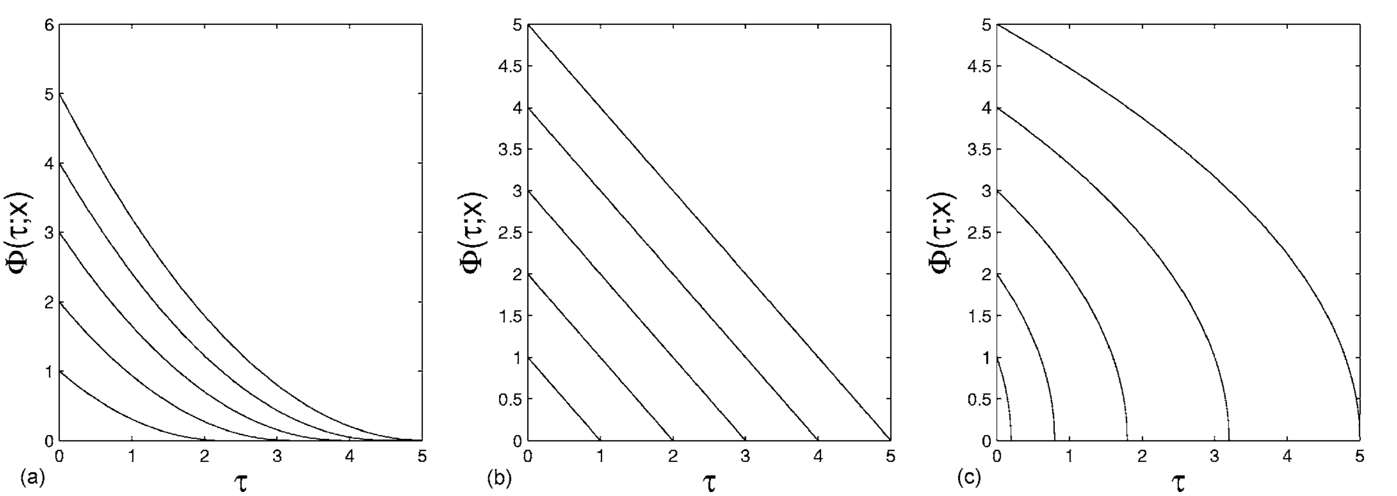

FIG. 4. Decay patterns of nonlinear shot noise systems with decay-dynamics governed by the ODE $\dot{X}=-(1/p)(X/5)^{1-p}$ (p being a positive-valued parameter). (a) $p=0.5$; (b) $p=1$; and (c) $p=2$. The decay patterns are finite-ranged and convex for $p<1$; linear for $p=1$; and concave for $p>1$. Note that the greater the initial shot magnitude, the longer the "lifetime" of the resulting decay pattern.

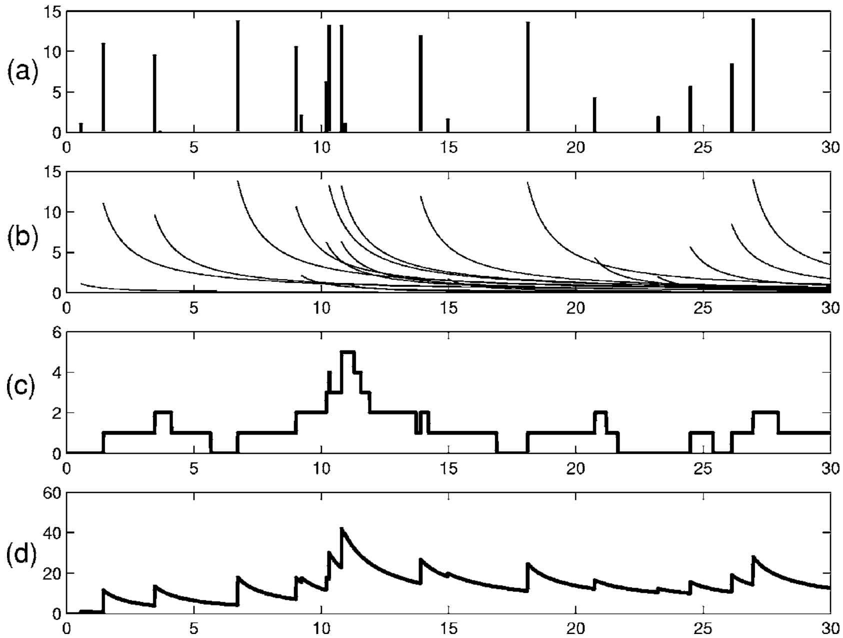

FIG. 5. A schematic illustration of shot noise systems with infinite-ranged and convex decay patterns. (a) The impact coordinates of incoming shots—the positions and heights of the spikes representing, respectively, the impact epochs and initial magnitudes of the incoming shots. (b) The resulting collection process Ξ—superimposing the decay patterns of the incoming shots. (c) The lth level process N_l (corresponding to the level $l=3$)—keeping count of the number of shot magnitudes above the level l. (d) The noise process ξ—aggregating-up the shot magnitudes.

Equations (7) and (8) imply that the Poissonian autocovariance function *characterizes* the *process-distribution* of the level processes $\{N_l\}_{l>0}$. The proof of these equations is analogous, in principle, to the proof of Proposition 4 in [16] (and is beyond the scope of this paper).

IV. SHOT NOISE STATISTICS

Based on the results of the previous section, we are now in position to analyze the shot noise's stationary structure (amplitudal behavior) and correlation structure (temporal behavior).

A. Stationary structure and the Noah effect

The stationary structure of the noise process ξ is governed and quantified by the Poissonian mean $\mu(\cdot)$: The noise process ξ is divergent (nonsummable) if $\int_0^1 \mu(l)dl = \infty$, and is convergent (summable) if $\int_0^1 \mu(l)dl < \infty$.

In case the noise process ξ is convergent then:

(1) The Laplace transform of the noise's stationary distribution is given by

$$\mathbf{E}[\exp\{-\theta\xi(t)\}] = \exp\left\{-\theta\int_0^\infty \exp\{-\theta l\}\mu(l)dl\right\} \quad (\theta \geq 0). \tag{9}$$

(2) The mth cumulant of the noise's stationary distribution is given by

$$\mathbf{E}[\xi(t)^m] = m\int_0^\infty l^{m-1}\mu(l)dl \quad (m = 1,2,\ldots). \tag{10}$$

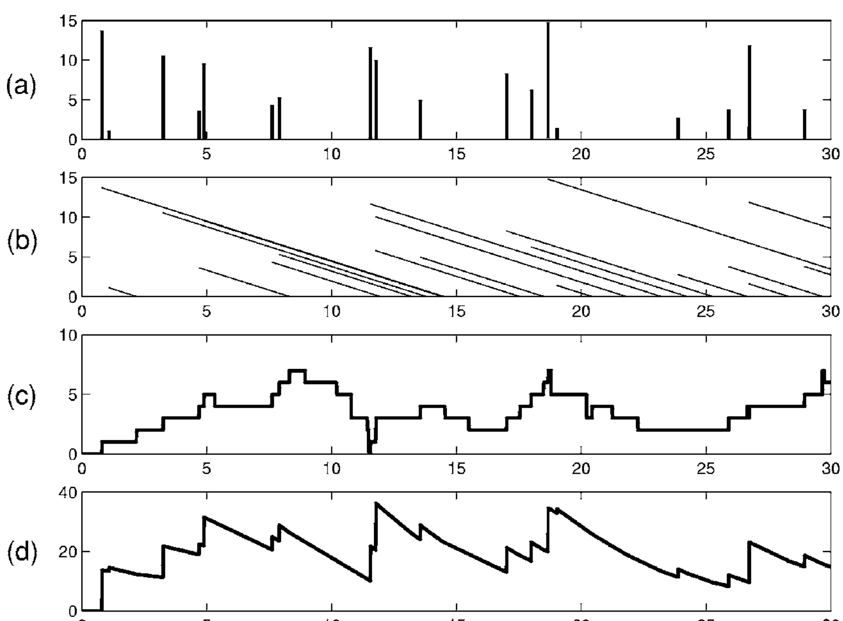

FIG. 6. A schematic illustration of shot noise systems with finite-ranged and linear decay patterns. (a) The impact coordinates of incoming shots—the positions and heights of the spikes representing, respectively, the impact epochs and initial magnitudes of the incoming shots. (b) The resulting collection process Ξ—superimposing the decay patterns of the incoming shots. (c) The lth level process N_l (corresponding to the level $l=3$)—keeping count of the number of shot magnitudes above the level l. (d) The noise process ξ—aggregating-up the shot magnitudes.

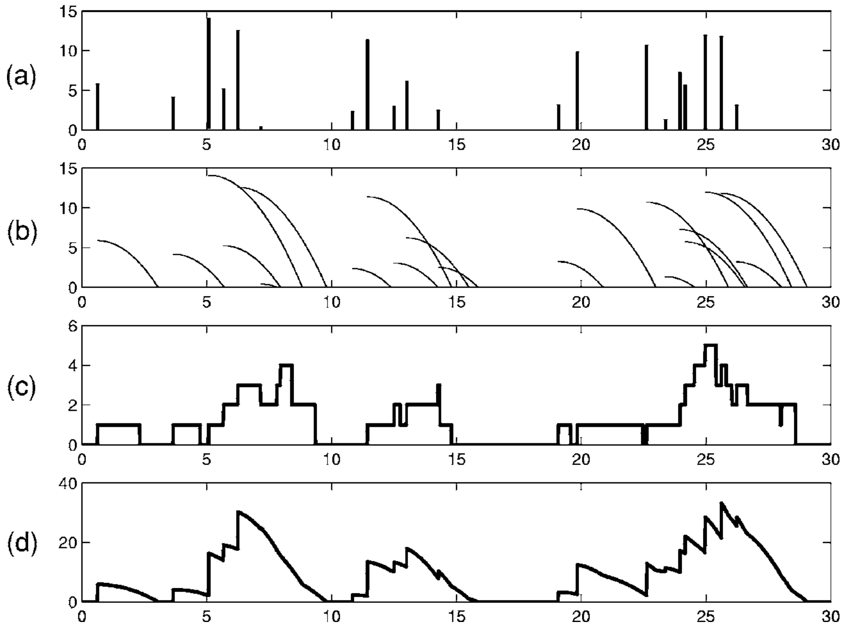

FIG. 7. A schematic illustration of shot noise systems with finite-ranged and concave decay patterns. (a) The impact coordinates of incoming shots—the positions and heights of the spikes representing, respectively, the impact epochs and initial magnitudes of the incoming shots. (b) The resulting collection process Ξ—superimposing the decay patterns of the incoming shots. (c) The lth level process N_l (corresponding to the level $l=3$)—keeping count of the number of shot magnitudes above the level l. (d) The noise process ξ—aggregating-up the shot magnitudes.

The above-stated results are consequences of Eq. (5) and Campbell's theorem—a key result in the theory of Poisson processes (see Sec. III B in [18]).

The Noah effect

A stationary random process is said to display the *Noah effect* if its stationary distribution is *heavy-tailed* [19]. Loosely speaking, this means that the probability tail of its stationary distribution is, asymptotically, a power law. The precise definition is that the probability tail of its stationary distribution is *regularly varying*—a mathematical notion described in the Appendix (see Sec. A 3).

The etymology of the term "*Noah effect*," coined by Mandelbrot and Wallis [8], stems from the biblical story of Noah's great flood: "…were all the fountains of the great deep broken up, and the windows of heaven were opened. And the rain was upon the earth forty days and forty nights." Genesis, 7: 11-12.

In case the noise process ξ is convergent, there is an intimate connection between the Noah effect and the Poissonian mean $\mu(\cdot)$, given by the following result. The probability tail of the noise's stationary distribution is regularly varying of order $\nu=-\alpha$ $(\alpha>0)$ if and only if the Poissonian mean is such, in which case

$$\mathbf{P}(\xi(t) > l) \underset{l\to\infty}{\sim} \mu(l). \tag{11}$$

Equation (11) follows from Eq. (9) due to Theorem 8.2.1 in [20]. As a specific example consider a Poissonian mean satisfying the asymptotics $\mu(l)\sim al^{-\alpha}$ (as $l\to\infty$), where the coefficient a and the exponent α are positive parameters. In this case the stationary noise level is of (i) finite mean if and only if $\alpha>1$; and (ii) finite variance if and only if $\alpha>2$.

B. Correlation structure and the Joseph effect

The most common and straightforward approach to investigate a given process' correlation structure is via its autocovariance function. However, problems arise when trying to do so in case of the noise process ξ. First, the noise's stationary distribution may fail to possess a finite variance—rendering the autocovariance function undefined. Worse so, the noise may even fail to converge—rendering the very noise process ξ undefined.

Nonetheless, the noise's level processes $\{N_l\}_{l>0}$ are always well-defined, and their Poissonian autocovariance function has three major advantages over the "standard" autocovariance function of the noise process ξ: (i) it always exists—no matter how "wild" the amplitudal fluctuations of the shot noise under consideration are; (ii) it *characterizes* the process-distribution of the shot noise under consideration—rather than being merely a second-order statistic; and (iii) it provides *resolution*—a "feature" which we now elaborate on.

The autocovariance function of the noise process ξ—if it exists—measures the statistical correlation between the noise levels $\xi(t)$ and $\xi(t+\tau)$. That is, it gives a single numerical value for the "overall measure-of-correlation" between the *aggregates* $\xi(t)$ and $\xi(t+\tau)$, without dwelling into their *composites*. The Poissonian autocovariance, on the other hand, provides a more detailed quantitative assessment of the temporal correlations of the noise process ξ by means of the resolution parameter l. The Poissonian autocovariance "pokes" about the underlying structure of the noise aggregates $\xi(t)$ and $\xi(t+\tau)$—namely, the shot collections $\Xi(t)$ and $\Xi(t+\tau)$—and measures the correlation of their composites, after sifting them according to their size.

The Joseph effect

A stationary random process is said to display the *Joseph effect* if its autocovariance function is *long-range dependent* [21–23]. Loosely speaking, this means that the autocovariance is, asymptotically, a power law. The precise definition is that the autocovariance is *regularly varying*—a mathematical notion described in the Appendix (see Sec. A 3).

The etymology of the term "Joseph effect," coined by Mandelbrot and Wallis [8], stems from the biblical story of Joseph's prophesy: "…there came seven years of great plenty throughout the land of Egypt. And there shall arise after them seven years of famine…" Genesis, 41: 29-30.

The following result, regarding the Joseph effect, holds. If the function $\Lambda(\Psi(\cdot;l))$ is regularly varying of order $\nu=-(1+\beta)$ $(\beta>0)$ then the Poissonian autocovariance function is regularly varying of order $\nu=-\beta$, and

$$r_l(\tau) \underset{\tau\to\infty}{\sim} \frac{\tau}{\beta}\Lambda(\Psi(\tau;l)). \tag{12}$$

The proof of Eq. (12) is given in the Appendix (see Sec. A 3).

C. Simultaneous display of the Noah and Joseph effects

We give three examples illustrating the *simultaneous* display of the Noah and Joseph effects. In the examples, the shot-inflow is taken to be *scale-invariant*—governed by the power-law Poissonian rate function $\Lambda(l)=al^{-\alpha}$ $(a,\alpha>0)$. The decay mechanisms are taken to be of a power-law form with exponent β and coefficient b $(b,\beta>0)$.

(In all three examples the coefficients a and b are chosen—with no loss of generality—so that to yield coefficient-free results. The precise derivations involve standard asymptotic calculations of integrals, and are omitted.)

(1) Linear shot noise systems. Consider a linear shot noise system with impulse-response function $h(\tau)=(1+b\tau)^{-\beta}$. The resulting Poissonian mean and Poissonian autocorrelation are given, respectively, by

$$\mu(l) = \frac{1}{l^{\gamma}} \text{ and } \rho_l(\tau) = \frac{1}{\tau^{\delta}}, \tag{13}$$

where $\gamma=\alpha$ and $\delta=\alpha\beta-1$. (In this example it is required that the exponents α and β satisfy the condition $\alpha\beta>1$.)

(2) $M/G/\infty$-type shot noise systems. Consider an $M/G/\infty$-type shot noise system with draining function $g(\tau)=b\tau^{\beta}$. The resulting Poissonian mean and Poissonian autocorrelation are given, respectively, by

$$\mu(l) = \frac{1}{l^{\gamma}} \text{ and } \rho_l(\tau) \underset{\tau\to\infty}{\sim} \frac{l^{\gamma}}{\tau^{\delta}}, \tag{14}$$

where $\gamma=\alpha-1/\beta$ and $\delta=\alpha\beta-1$. (In this example it is required that the exponents α and β satisfy the condition $\alpha\beta>1$.)

(3) Nonlinear shot noise systems. Consider a nonlinear shot noise system with decay-dynamics governed by the ODE $\dot{X}=-bX^{1-\beta}$. The resulting Poissonian mean and Poissonian autocorrelation are given, respectively, by

$$\mu(l) = \frac{1}{l^{\gamma}} \text{ and } \rho_l(\tau) \underset{\tau\to\infty}{\sim} \frac{l^{\gamma}}{\tau^{\delta}}, \tag{15}$$

where $\gamma=\alpha-\beta$ and $\delta=\alpha/\beta-1$. (In this example it is required that the exponents α and β satisfy the condition $\alpha>\beta$.)

In all three examples the noise process ξ is convergent (summable) if and only if the exponent γ is in the range $0<\gamma<1$—in which case Eq. (9) implies that the system's stationary noise level is governed by a γ*-stable Lévy distribution*:

$$\mathbf{E}[\exp\{-\theta\xi(t)\}] = \exp\{-\Gamma(1-\gamma)\theta^{\gamma}\} \tag{16}$$

$(\theta\geq 0)$.

D. Amplitudal-temporal decoupling

In the first example of the previous section (linear shot noise systems), the Poissonian autocorrelation $\rho_l(\cdot)$ turned out to be independent of the resolution variable l. In other words, the Poissonian autocovariance function admitted the *amplitudal-temporal factorization*

$$r_l(\tau) = F(\tau;l) = \mu(l)\rho(\tau), \tag{17}$$

where $\rho(\tau)$ is the resolution-independent Poissonian autocorrelation [and $\mu(l)$ is the "lag-independent" Poissonian mean].

Can the cases in which such an amplitudal-temporal factorization take place be characterized? The answer, for the three classes of shot noise system-models presented above, is affirmative (the parameters a and α appearing below are positive).

(1) Linear shot noise systems [with impulse-response function $h(\cdot)$]. An amplitudal-temporal factorization holds if and only if the Poissonian rate function admits the power-law form $\Lambda(l)=al^{-\alpha}$, in which case

$$\mu(l) = c\Lambda(l) \text{ and } \rho(\tau) = \frac{1}{c}\int_{\tau}^{\infty} h(s)^{\alpha} ds, \tag{18}$$

where $c=\int_0^{\infty}h(s)^{\alpha}ds$ (provided that this integral is convergent). The Poissonian correlation structure of such systems can be reverse-engineered: in order to obtain a system with a "target" correlation function $\rho(\cdot)$ the impulse-response function should be set to be $h(\cdot)=[-\rho'(\cdot)]^{1/\alpha}$.

(2) $M/G/\infty$-type shot noise systems [with draining function $g(\cdot)$]. An amplitudal-temporal factorization holds if and only if the Poissonian rate function admits the exponential form $\Lambda(l)=a\exp\{-\alpha l\}$, in which case

$$\mu(l) = c\Lambda(l) \text{ and } \rho(\tau) = \frac{1}{c}\int_{\tau}^{\infty} \exp\{-\alpha g(s)\} ds, \tag{19}$$

where $c=\int_0^{\infty}\exp\{-\alpha g(s)\}ds$ (provided that this integral is convergent). The Poissonian correlation structure of such systems can be reverse-engineered: in order to obtain a system with a "target" correlation function $\rho(\cdot)$ the draining function should be set to be $g(\cdot)=\frac{-1}{\alpha}\ln(-\rho'(\cdot))$.

(3) Nonlinear shot noise systems [with ODE $\dot{X}=-v(X)$ governing the shots' decay-dynamics]. [The function $v(\cdot)$ being positive-valued on the positive half-line, and such that the ODE it induces admits unique solutions.] Set $V(\cdot)$ to be a primitive of the function $1/v(\cdot)$ [namely, $V'(\cdot)=1/v(\cdot)$]. An amplitudal-temporal factorization holds if and only if the Poissonian rate is of the form $\Lambda(l)=a\exp\{-\alpha V(l)\}$, in which case

$$\mu(l) = \frac{1}{\alpha}\Lambda(l) \text{ and } \rho(\tau) = \exp\{-\alpha\tau\}. \quad (20)$$

The Poissonian amplitudal structure of such systems can be reverse-engineered: in order to obtain a system with a "target" mean function $\mu(\cdot)$ the ODE should be set to be governed by the function $v(\cdot) = -\alpha\mu(\cdot)/\mu'(\cdot)$.

The proof of Eqs. (18)–(20) is given in the Appendix (see Sec. A 4).

V. THE POISSONIAN AUTOCORRELATION

The Poissonian autocorrelation $\rho_l(\cdot)$ has three interpretations. They are

(1) the *autocorrelation* function of the level processes;
(2) a Poissonian "*splitting ratio*" of the noise's shot-magnitudes; and
(3) a "*survival probability*" of the noise's shot-magnitudes.

The first interpretation was discussed above. We now describe the two other interpretations. In what follows consider—with no loss of generality—the shot noise system at the time epochs $t=0$ and $t=\tau$, and call shot-magnitudes above the resolution level l "observable."

A. Splitting ratio

Shots observable at time τ are of two types: (i) "old shots" originating in shots hitting the system *up to* 0; and (ii) "new shots" originating in shots hitting the system *after* time 0 (and up to time τ). Hence

$$N_l(\tau) = N_l^{\text{old}}(\tau) + N_l^{\text{new}}(\tau), \quad (21)$$

where $N_l^{\text{old}}(\tau)$ and $N_l^{\text{new}}(\tau)$ are, respectively, the counts of the old and the new shots observable. See Fig. 8 for a schematic illustration of "old" and "new" shots (with no loss of generality, Fig. 8 depicts the case of linear decay patterns).

The counts $N_l^{\text{old}}(\tau)$ and $N_l^{\text{new}}(\tau)$ are independent random variables, and

(1) $N_l^{\text{old}}(\tau)$ is *Poisson-distributed* with mean $\rho_l(\tau)\mu(l)$ and
(2) $N_l^{\text{new}}(\tau)$ is *Poisson-distributed* with mean $[1-\rho_l(\tau)]\mu(l)$.

The proof of this decomposition result is analogous, in principle, to the proof of Proposition 3 in [14] (and is beyond the scope of this paper).

The decomposition result implies that exactly a $\rho_l(\tau)$-proportion of the Poissonian intensity of shots observable at time τ should be attributed to shots observable at time 0. Hence the Poissonian autocorrelation $\rho_l(\tau)$ is a Poissonian "*splitting ratio*" which quantitatively measures the influence the system's "present state" (at time 0) has on the system's "future state" (at time τ).

B. Survival probability

Assume that at time 0 there are exactly n observable shots. Shots observable at time τ are of two types: (i) "*surviving shots*" originating from shots observable at time 0 which remained observable until time τ; and (ii) "*new shots*" originating from shots hitting the system after time 0 (and up to time τ). Hence

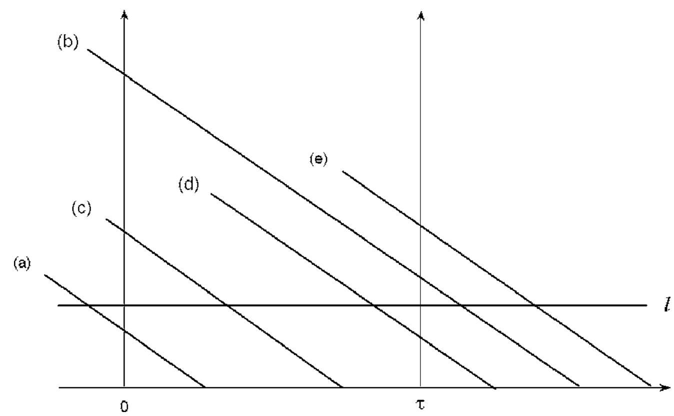

FIG. 8. A schematic illustration of "old," "new," and "surviving" shots (depicted, with no loss of generality, for shot noise systems with linear decay patterns). Shots (a)–(c) are "old shots"—arriving at the system up to time 0. Shots (d) and (e) are "new shots"—arriving at the system between times 0 and τ. Shots (b) and (c) are observable at time 0. Shots (b) and (e) are observable at time τ. Shot (b) is the only "surviving shot"—arriving before time 0 and remaining observable throughout the time interval $[0,\tau]$.

$$N_l(\tau)|_{N_l(0)=n} = N_l^{\text{sur}}(\tau) + N_l^{\text{new}}(\tau), \quad (22)$$

where $N_l^{\text{sur}}(\tau)$ and $N_l^{\text{new}}(\tau)$ are, respectively, the counts of the surviving shots and the new shots observable. See Fig. 8 for a schematic illustration of "surviving" and "new" shots (with no loss of generality, Fig. 8 depicts the case of linear decay patterns).

The counts $N_l^{\text{sur}}(\tau)$ and $N_l^{\text{new}}(\tau)$—given the information $\{N_l(0)=n\}$—are independent random variables, and

(1) $N_l^{\text{sur}}(\tau)$ is *binomially distributed* with parameters n and $p=\rho_l(\tau)$ and
(2) $N_l^{\text{new}}(\tau)$ is *Poisson distributed* with mean $[1-\rho_l(\tau)]\mu(l)$.

The proof of this conditional decomposition result is analogous, in principle, to the proof of Proposition 3 in [16] (and is beyond the scope of this paper).

The conditional decomposition result implies that each shot observable at time 0—independently of all other shots observable at time 0—has probability $p=\rho_l(\tau)$ of remaining observable throughout the time interval $[0,\tau]$. Hence the Poissonian autocorrelation function $\rho_l(\cdot)$ is the "survival probability" of shot magnitudes above the resolution level l.

If we set $\mathcal{R}_l$ to be the "residual lifetime" of shot-magnitudes above the resolution level l, then

$$\mathbf{P}(\mathcal{R}_l > \tau) = \rho_l(\tau) \quad (23)$$

$(\tau \geqslant 0)$. Equation (23) yields an interesting "Noah interpretation" of the Joseph effect in shot noise systems.

The Poissonian autocorrelation $\rho_l(\cdot)$ is *long-range depen-*

dent if and only if the distribution of the residual lifetime $\mathcal{R}_l$ is *heavy tailed*.

VI. CONCLUSIONS

Shot noise processes possess an underlying Poissonian structure. Exploiting this fundamental structure, we introduced the *Poissonian autocovariance* function. The Poissonian autocovariance has three major advantages over the "standard" autocovariance: (i) it is always well-defined—no matter how "wild" the shot noise's amplitudal fluctuations are; (ii) it *characterizes* the shot noise's process-distribution, and hence, in particular, it characterizes the shot noise's stationary structure (amplitudal behavior) and correlation structure (temporal behavior); and (iii) it provides *resolution*—facilitating a resolution-contingent analysis of the shot noise under consideration.

The Poissonian autocovariance induced two other functions: the *Poissonian mean* and the *Poissonian autocorrelation*. The Poissonian mean governs and quantifies the shot noise's amplitudal structure. The Poissonian autocorrelation governs and quantifies the shot noise's correlation structure and has three different interpretations: (i) it is the autocorrelation function of the shot noise's level processes; (ii) it is a Poissonian "splitting ratio" quantifying the shot noise's temporal dependencies; and (iii) it is the "survival probability" of shot-magnitudes above a given resolution level.

The "Poissonian methodology" developed enables a precise quantitative analysis of general shot noise processes at large. In particular, it enables the quantitative analysis of shot noise processes displaying simultaneously the Noah and Joseph effects.

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APPENDIX

1. The $M/G/\infty$ queuing system

The most fundamental model of an infinite-server queuing system is the so-called $M/G/\infty$ system. In the standard nomenclature of queuing theory, the "acronym" $M/G/\infty$ is a shorthand notation for a queuing system with: Markovian arrival of jobs (M); general distribution of job-sizes (G); and an infinite number of servers (∞). The $M/G/\infty$ system was proposed in [24] as a mathematical model for a textile yarn, and was further investigated in [25]. For a recent exposition and study of the $M/G/\infty$ system the readers are referred to [26].

In the $M/G/\infty$ queuing system jobs of random sizes arrive, following a standard Poisson process, at a service facility with infinitely many servers. Each job, upon arrival, is immediately attended by a server. Servers operate at a deterministic unit rate—processing one work-unit per one time-unit. Jobs leave the system upon service completion. Hence (i) the "lifetime" of a job in the system equals its initial size; and (ii) the residual service required by a job of initial size x, after having spent τ units of time in the system, is $\max\{x-\tau,0\}$.

The $M/G/\infty$ system's *workload* at time t is defined as the aggregate of the residual service-times of all jobs present in the system (at time t). The $M/G/\infty$ workload process is thus a shot noise process with decay pattern $\Phi(\tau;x)=\max\{x-\tau,0\}$.

The $M/G/\infty$ system's *queue* at time t, denoted $Q(t)$, is defined as the number of jobs present in the system (at time t). If $\Lambda(l)$ is the Poissonian arrival rate of jobs whose size is greater than the level l ($l>0$) then ([26], Proposition 1):

(1) The random variable $Q(t)$ is Poisson-distributed with mean

$$\mathbf{E}[Q(t)] = \int_0^\infty \Lambda(l)dl. \tag{A1}$$

(2) The autocovariance function of the queue process $Q=[Q(t)]_t$ is given by

$$\mathrm{Cov}[Q(t),Q(t+\tau)] = \int_\tau^\infty \Lambda(l)dl \tag{A2}$$

($\tau\geqslant 0$).

2. Mean and autocovariance of the level processes: Proofs

Consider a general shot noise system with Poissonian rate function $\Lambda(\cdot)$ and decay pattern $\Phi(\cdot;\cdot)$ [inverse decay pattern $\Psi(\cdot;\cdot)$], and fix a resolution level l ($l>0$).

Let $\mathcal{L}_l(x)$ denote the "lifetime" of a shot, with initial shot magnitude x, above the resolution level l. The properties of the decay pattern imply that

$$\{\mathcal{L}_l(x) > s\} \Leftrightarrow \{\Phi(s;x) > l\} \Leftrightarrow \{x > \Psi(s;l)\}$$

($s\geqslant 0$). Hence shots which remain above the resolution level l for over s time-units arrive at the system according to the Poissonian rate

$$\Lambda_l(s) := \Lambda(\Psi(s;l)) \tag{A3}$$

($s\geqslant 0$).

Now, observe that the level process N_l—tracking the number of shot-magnitudes above the resolution level l—is identical, in law, to the queue process Q of an $M/G/\infty$ queuing system with Poissonian arrival rate $\Lambda_l(\cdot)$. Thus substituting Eq. (A3) into Eqs. (A1) and (A2) yields, respectively, Eqs. (2) and (3).

3. Regular variation

A function $f(x)$ ($x\geqslant 0$) is *regularly varying* of order ν (ν real) if

$$\lim_{x\to\infty} f(cx)/f(x) = c^\nu$$

holds for all positive c [20].

Slowly varying functions are regularly varying functions of order 0. The class of slowly varying functions includes constants, logarithms, iterated logarithms, and powers of logarithms.

A function $f(\cdot)$ is regularly varying of order ν if and only if $f(x) \sim x^{\nu} \cdot f_0(x)$ (as $x \to \infty$), where $f_0(\cdot)$ is a slowly varying function. Thus, loosely speaking, regularly varying functions can be considered as "asymptotic power-laws."

Proof: Equation (12). Equation (12) is a special case of the following general result: If the function $f(\cdot)$ is regularly varying of order $\nu = -(1+\beta)$ ($\beta > 0$) then the function $f_1(x) = \int_x^\infty f(x')dx'$ ($x \geqslant 0$) is regularly varying of order $\nu = -\beta$, and

$$f_1(x) \underset{x\to\infty}{\sim} \frac{1}{\beta} x f(x).$$

Indeed, using L'Hospital's rule and the regular variation of the function $f(\cdot)$ yields:

$$\lim_{x\to\infty} \frac{f_1(cx)}{f_1(x)} = \lim_{x\to\infty} \frac{f(cx)c}{f(x)} = c^{-\beta};$$

and

$$\lim_{x\to\infty} \frac{f_1(x)}{\frac{1}{\beta} x f(x)} = \beta \lim_{x\to\infty} \int_1^\infty \frac{f(sx)}{f(x)} ds = \beta \int_1^\infty s^{-(1+\beta)} ds = 1.$$

4. Amplitudal-temporal decoupling: Proofs

Combining Eqs. (1) and (17) together, and differentiation with respect to the temporal variable τ, yields:

$$\rho'(\tau) = \frac{\Lambda(\Psi(\tau;l))}{\mu(l)}. \tag{A4}$$

In the case of linear shot noise systems [with impulse-response function $h(\cdot)$] we have $\Psi(\tau;l) = l/h(\tau)$. Hence Eq. (A4) holds if and only if the Poissonian rate function $\Lambda(\cdot)$ is a power law. This, in turn, implies Eq. (18).

In the case of $M/G/\infty$-type shot noise systems [with draining function $g(\cdot)$] we have $\Psi(\tau;l) = l + g(\tau)$. Hence Eq. (A4) holds if and only if the Poissonian rate function $\Lambda(\cdot)$ is an exponential. This, in turn, implies Eq. (19).

In the case of nonlinear shot noise systems [with ODE $\dot{X} = -v(X)$ governing the shots' decay-dynamics] we have $\Psi(\tau;l) = V^{-1}[V(l) + \tau]$, where $V(\cdot)$ is a primitive of the function $1/v(\cdot)$ [namely, $V'(\cdot) = 1/v(\cdot)$], and where $V^{-1}(\cdot)$ is the inverse of the function $V(\cdot)$. [The function $v(\cdot)$ being positive-valued on the positive half-line, and such that the ODE it induces admits unique solutions.] Hence Eq. (A4) holds if and only if the composite function $(\Lambda \circ V^{-1})(\cdot)$ is an exponential. This, in turn, implies Eq. (20).

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